Technical Report: The Homogeneous Second-Order Descent Framework with Inexact Eigenvalue Computations

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Abstract

This report is a companion to "Homogeneous Second-Order Descent Framework: A Fast Alternative to Newton-Type Methods" [3]. In this report, we show how to allow inexactness in the subproblems.

Contents

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1 Introduction

Recently, Zhang et al. [7] proposed a homogeneous second-order descent method (HSODM) unconstrained smooth optimization problem $\min_x f(x)$. The method is inspired by the classic trick in quadratic programming [6, 5, 2]. In their method, the iterates are constructed by solving the *Ordinary* Homogeneous Model (OHM)

$$\min_{\|[v;t]\|\le 1} \psi_k(v,t;F_k) := [v;t]^T F_k [v;t] \text{ with the aggregated matrix } F_k := \begin{bmatrix} H_k & g_k \\ g_k^T & \delta \end{bmatrix}, \quad (1.1)$$

where variables $v \in \mathbb{R}^n, t \in \mathbb{R}$ and $\delta \le 0$ is a prescribed parameter. The method sets the control term δ such that the $[v;t]$ corresponds to the leftmost eigenvector of the aggregated matrix F_k. Then the HSODM undergoes a simple strategy to find a step size η_k, e.g., by a line-search method, and updates the iterate as $x_{k+1} = x_k + \eta_k(v_k/t_k)$ using the direction $[v_k;t_k]$ generated by OHM. A specialized Lanczos method is used to solve the subproblem based on OHM. Combining with a line-search strategy for η_k, the original HSODM needs $O(n^3\epsilon^{-3/2})$ (exact eigenvalues) and $O(n^2\epsilon^{-7/4})$ (inexact eigenvalues) arithmetic operations to find second-order stationary points nonconvex problems.

In He et al. [3], we extend the idea of homogenization and introduce the *Generalized* Homogeneous Model (GHM)

$$\min_{\|[v;t]\|\le 1} \psi_k(v,t;F_k) := [v;t]^T F_k [v;t] \quad \text{with} \quad F_k := \begin{bmatrix} H_k & \phi_k(x_k) \\ \phi_k(x_k)^T & \delta_k \end{bmatrix}, \ \delta_k \in \mathbb{R}, \quad (1.2)$$

albeit now δ_k is allowed for some adaptiveness. Furthermore, we introduce the transformation $\phi_k : \mathbb{R}^n \mapsto \mathbb{R}^n$ in place of the gradient g_k. We show this flexibility facilitates a machinery to realize other second-order methods and, more importantly, a general homogeneous framework in which new algorithms can be designed.

The purpose of this note is to show how to allow inexactness in GHMs when implementing an *inexact adaptive HSODM*, to complement the convergence analysis in He et al. [3, Algorithm 2].

Notations We introduce the notations used throughout the paper. Denote the standard Euclidean norm in space $\mathbb{R}^n$ by $\|\cdot\|$. Let $B(x_c, r)$ denote the ball whose center is x_c and radius is r, i.e., $B(x_c,r) = \{x \in \mathbb{R}^n \mid \|x - x_c\| \le r\}$. For a matrix $A \in \mathbb{R}^{n\times n}$, $\|A\|$ represents the induced $\mathcal{L}_2$ norm. Let $A^\star$ denote the pseudo-inverse of the matrix A. We let $P_{\mathcal{X}}$ be the orthogonal projection operator onto a space, where $\mathcal{X} \subseteq \mathbb{R}^n$. We use mod to denote the binary modulo operation. We say a vector y is orthogonal to a subspace $\mathcal{S}$, i.e. $y \perp \mathcal{S}$ if for any nonzero vector $u \in \mathcal{S}$, $u^T y = 0$.

Next, we introduce the following notations for eigenvalues of Hessian H_k. At an iterate of the algorithm x_k, we assume H_k has d $(1 \le d \le n)$ distinct eigenvalues $\{\lambda_1(H_k), ..., \lambda_d(H_k)\}$

where $\lambda_1(H_k) < ... < \lambda_d(H_k)$ and $\mathcal{S}_1(H_k), ..., \mathcal{S}_d(H_k)$ are subspaces spanned by corresponding eigenvectors. We sometimes use $\lambda_{\min}, \lambda_{\max}$ as synonyms for λ_1 and λ_d, respectively. We denote the condition number of H_k as $\kappa(H_k) = \frac{\lambda_d(H_k)}{\lambda_1(H_k)}$. Since the discussion on eigenvalues is mostly restricted at the iterate x_k only, we sometimes drop the index k for simplicity.

2 Basic Results

We present the basic results for homogeneous systems. Most materials are covered in Zhang et al. [7] and He et al. [3]. We first present the properties if the eigenvalue is computed exactly.

Lemma 2.1 (Optimality condition). *$[v_k; t_k]$ is the optimal solution of the subproblem (1.2) if and only if there exists a dual variable $\theta_k \geq 0$ such that*

$$\begin{bmatrix} H_k + \theta_k \cdot I & \phi_k \\ \phi_k^T & \delta_k + \theta_k \end{bmatrix} \succeq 0, \tag{2.1}$$

$$\begin{bmatrix} H_k + \theta_k \cdot I & \phi_k \\ \phi_k^T & \delta_k + \theta_k \end{bmatrix} \begin{bmatrix} v_k \\ t_k \end{bmatrix} = 0, \tag{2.2}$$

$$\theta_k \cdot (\|[v_k; t_k]\| - 1) = 0. \tag{2.3}$$

The following Lemma describes the upper bound for θ_k.

Lemma 2.2 (Upper bound of θ_k). *In GHM, it holds that:*

$$\theta_k \leq \max\{-\delta_k, -\lambda_1(H_k), 0\} + \|\phi_k(x)\|. \tag{2.4}$$

Next, we move to the case where $t_k = 0$. Let us recall the following lemmas on the spectrum of F_k if $\phi_k \perp \mathcal{S}_1(H_k)$.

Lemma 2.3 (Lemma 3.1, 3.2, Rojas et al. [4]). *For any $q \in \mathcal{S}_j(H_k), 1 \leq j \leq d$, define*

$$p_j = -\left(H_k - \lambda_j(H_k)I\right)^{\star} \phi_k,\ \tilde{\alpha}_j = \lambda_j(H_k) - \phi_k^T p_j,$$

then

(a) $(\lambda_j(H_k), [0; q])$ *is an eigenpair of F_k if and only if $\phi_k \perp \mathcal{S}_j(H_k)$.*

(b) $(\lambda_j(H_k), [1; p_j])$ *is an eigenpair of F_k if and only if $\phi_k \perp \mathcal{S}_j(H_k)$ and $\delta_k = \tilde{\alpha}_j$.*

In He et al. [3], we introduce the concept of *auxiliary functions* to facilitate the discussion of homogeneous algorithms. These functions are defined as the following.

Definition 2.1 (Auxiliary functions of δ_k)**.** *At each iterate x_k, consider the GHM with δ_k and let v_k, t_k be the corresponding solution. $\overline{\Delta}_k < +\infty$ is an upper bound for the step.*

$$\Delta_k : \mathbb{R} \mapsto \mathbb{R}_+,\ \Delta_k(\delta_k) := \begin{cases} \|v_k/t_k\|^2 & \textit{if } \delta_k < \tilde{\alpha}_1 \\ \overline{\Delta}_k & \textit{o.w.} \end{cases}$$
$$\omega_k : \mathbb{R} \mapsto \mathbb{R}_+,\ \omega_k(\delta_k) := \theta_k^2$$
$$h_k : \mathbb{R} \mapsto \mathbb{R}_+,\ h_k(\delta_k) := \frac{\omega_k(\delta_k)}{\Delta_k(\delta_k)}$$

He et al. [3] show that for $\phi_k \neq 0$, θ_k, ω_k are decreasingly convex and continuous for all $\delta_k \in \mathbb{R}$. If further $t_k \neq 0$, θ_k is differentiable such that $\frac{\mathrm{d}}{\mathrm{d}\delta_k}\theta_k = -\frac{1}{\Delta_k+1}$. Furthermore, Δ_k is continuous for all $\delta_k \in \mathbb{R}$. This implies the following result.

Lemma 2.4. *If $\phi_k \perp \mathcal{S}_1(H_k)$, $h_k(\delta)$ is discontinuous at $\tilde{\alpha}_1$; otherwise, $h_k(\delta)$ is continuous. Moreover, $h_k(\delta)$ is differentiable in δ is monotone decreasing.*

Based on these results, we can design a bisection procedure to find the some δ_k to locate h_k at some interval.

2.1 Using inexact eigenvalues

When using inexact eigenvalues, all mentioned quantities

$$(\theta_k, v_k, t_k, h_k)$$

are in approximate form to some extent. Under the Lanczos method, we denote $\gamma_k, [\hat{v}_k; \hat{t}_k]$ as Ritz pair that approximates leftmost eigen pair. The Lanczos method is terminated when the error tolerance $e_k := \theta_k - \gamma_k$ is sufficiently small, e.g., $e_k \leq O(\sqrt{\epsilon})$. The following lemma from Zhang et al. [7] describes the approximate optimal condition.

Lemma 2.5 (Approximate Optimal Conditions)**.** *If the Lanczos method is used to solve* (3.3) *such that $e_k := \theta_k - \gamma_k$, then*

$$H_k\hat{v}_k + \phi_k\hat{t}_k + \gamma_k\hat{v}_k = r_k \tag{2.5a}$$
$$\phi_k^T\hat{v}_k + \delta_k\hat{t}_k + \gamma_k\hat{t}_k = \sigma_k \tag{2.5b}$$
$$F_k + (\gamma_k + e_k)I \succeq 0 \tag{2.5c}$$
$$[\hat{v}_k; \hat{t}_k] \perp [r_k; \sigma_k], \tag{2.5d}$$

where $[r_k; \sigma_k]$ are the Ritz error.

We first discuss the case $\hat{t}_k \neq 0$, as an analog of the hard case is identified if $\hat{t}_k = 0$ When the subproblem is solved inexactly, the knowledge of the exact pair $(\theta_k, [v_k; t_k])$ is not present.

Hence both the auxiliary function $h_k(\delta_k)$ and model function $m_k(d)$ can only be approximated in terms of $\gamma_k, [\hat{v}_k; \hat{t}_k]$. These can be defined as the following:

$$\hat{h}_k(\delta_k) = \left(\frac{\gamma_k}{\|\hat{d}_k\|} \right)^2, \text{ with } \hat{d}_k = \hat{v}_k / \hat{t}_k, \tag{2.6}$$

and the cubic model as

$$\hat{m}_k(\hat{d}_k) = f(x_k) + \phi_k^T \hat{d}_k + \frac{1}{2} (\hat{d}_k)^T H_k \hat{d}_k + \frac{\sqrt{\hat{h}_k(\delta_k)}}{3} \|\hat{d}_k\|^3. \tag{2.7}$$

3 Global Convergence Analysis

Following [3], we present the complexity analysis of an inexact adaptive HSODM to find an ϵ-approximate second-order stationary point defined as follows.

Definition 3.1. *A point x is called an ϵ-approximate second-order stationary points if it satisfies the following conditions.*

$$\|\nabla f(x)\| \leq O(\epsilon) \tag{3.1a}$$

$$\lambda_1 \left(\nabla^2 f(x) \right) \geq \Omega(-\sqrt{\epsilon}). \tag{3.1b}$$

In addition, we consider a broad class of objective functions satisfying the second-order Lipschitz continuity.

Definition 3.2. *We call a function f has M-Lipschitz continuous Hessian if for all $x, y \in \mathbb{R}^n$,*

$$\|\nabla^2 f(x) - \nabla^2 f(y)\| \leq M \|x - y\|. \tag{3.2}$$

Now we are ready to present the adaptive HSODM for second-order Lipschitz continuous functions in Algorithm 1.

We further make the following assumption on function $\phi_k(x_k)$ at every iterate x_k in GHMs.

Assumption 3.1. *Suppose that there exists a uniform constant $\varsigma_\phi > 0$. Given an iterate $x_k \in \mathbb{R}^n$, if $t_k \neq 0$, then*

$$\|\phi_k(x_k) - g_k\| \leq \varsigma_\phi \|\hat{d}_k\|^2 \tag{3.4}$$

where $\hat{d}_k = \hat{v}_k / \hat{t}_k$.

We present the following stopping criterion to terminate a subproblem solver.

Condition 3.1 (Inexactness of the subproblem)**.** *Suppose that (3.3) is solved by the Lanczos method at a prefixed error tolerance $e_k \leq O(\epsilon^{1/2})$. The Lanczos method is terminated if the fol-*

Algorithm 1: The Adaptive HSODM

Initial point $x_0 \in \mathbb{R}^n$, $\delta_0 \in \mathbb{R}$, $I_h = \mathbb{R}$, $h_{\min} > 0$, parameter $0 < \iota_1 < \eta_2 < 1$, $\iota_1 > 1, \iota_3 \geq \iota_2 > 1, 0 < \iota_4 \leq 1, \sigma > 0$;

for $k = 0, 1, 2, \dots$ **do**

  $\phi_k = g_k, \delta_{k,0} = \delta_{k-1}$

  for $j = 0, 1, \dots, \mathcal{T}_k$ **do**

    Obtain the solution pair $(\gamma_{k,j},\ [\hat{v}_{k,j}; \hat{t}_{k,j}])$ of the subproblem

$$\min_{\|[v;t]\| \leq 1} \begin{bmatrix} v \\ t \end{bmatrix}^T \begin{bmatrix} \hat{h}_k & \phi_{k,j} \\ (\phi_{k,j})^T & \delta_{k,j} \end{bmatrix} \begin{bmatrix} v \\ t \end{bmatrix} \tag{3.3}$$

    if $\hat{t}_{k,j} = 0$ **then** `// check hard case, see Section 3.3`

      Go to Algorithm 2

      Break

    end

    Set $\hat{d}_{k,j} = \hat{v}_{k,j} / \hat{t}_{k,j}$, $\hat{h}_k(\delta_{k,j}) := \left(\gamma_{k,j} / \|\hat{d}_{k,j}\|\right)^2$;

    if $\sqrt{\hat{h}_k(\delta_k)} \in I_h$ *within tolerance* σ **then**

      Set $\hat{d}_k := \hat{d}_{k,j}$, $\delta_k = \delta_{k,j}$

      Break

    end

    Update $\delta_{k,j}$;

  end

  Compute

$$\rho_k := \frac{f(x_k + \hat{d}_k) - f(x_k)}{m_k(\hat{d}_k) - f(x_k)};$$

  if $\rho_k > \eta_2$ **then** `// very successful iteration`

    $I_h = \left[\max\left\{\sqrt{h_{\min}}, \iota_4 \sqrt{\hat{h}_k(\delta_k)}\right\}, \sqrt{\hat{h}_k(\delta_k)}\right]$, $x_{k+1} = x_k + \hat{d}_k$

  if $\iota_1 \leq \rho_k \leq \eta_2$ **then** `// successful iteration`

    $I_h = \left[\sqrt{\hat{h}_k(\delta_k)} / \iota_1, \iota_2 \sqrt{\hat{h}_k(\delta_k)}\right]$, $x_{k+1} = x_k + \hat{d}_k$

  else `// unsuccessful iteration`

    $I_h = \left[\iota_2 \sqrt{\hat{h}_k(\delta_k)}, \iota_3 \sqrt{\hat{h}_k(\delta_k)}\right]$, $x_{k+1} = x_k$

  end

end

lowing conditions hold:

$$\|\phi_k + (H_k + \sqrt{\hat{h}_k(\delta_k)}\|\hat{d}_k\| I)\hat{d}_k\| \leq \varsigma_r \|\hat{d}_k\|^2 \tag{3.5a}$$

$$\gamma_k + \delta_k \geq 0 \tag{3.5b}$$

$$\sqrt{h_{\min}}\|\hat{d}_k\| \geq e_k. \tag{3.5c}$$

Note that the inexact conditions above will be satisfied gradually as the Krylov subspace evolves. Here ς_r can be set as any positive constant, which measures the relative error of the first-order optimality condition in a similar manner to the conditions imposed in inexact Newton methods [1]. The condition (3.5b) requires that the Ritz value is at least as good as δ_k. Such a requirement is discussed in [7], by using a skewed randomized initialization in the Lanczos method. While we propose the last condition (3.5c), it is only required for convergence to second-order stationarity, which may be violated at the beginning of the algorithm.

Establishing the convergence of such inexact variant of [3, Algorithm 2] induces a set of key questions:

(a) How the inexactness propagates the descent lemmas.

(b) How to perform a valid bisection using $\hat{h}_k(\delta)$ instead of h_k with the same $O(\log(1/\sigma))$ convergence rate;

(c) How the perturbation idea works under inexact solutions as a correspondence to [3, Algorithm 3]

These questions will be answered in order in the following subsections.

3.1 Convergence analysis

For now, we assume that the hard case $\hat{t}_k = 0$ does not occur, and these issues will be addressed later. In the following, we show that the step will eventually become a successful step, with a large enough $\hat{h}_k$ and an upper bound of the auxiliary function $\hat{h}_k$.

Apparently, we introduce the following arguments for a descent step.

Lemma 3.1. *Suppose $\hat{d}_k$ satisfies (3.5b), then we have the following model decrease*

$$f(x_k) - \hat{m}_k(\hat{d}_k) \geq \frac{\sqrt{\hat{h}_k(\delta_k)}}{6}\|\hat{d}_k\|^3 \geq \Omega(\|\hat{d}_k\|^3). \tag{3.6}$$

Proof. Note that from (2.5) and (3.5b), we have

$$(\hat{d}_k)^T H_k \hat{d}_k + \gamma\|\hat{d}_k\|^2 + 2\phi_k^T \hat{d}_k = -\delta_k - \gamma_k \leq 0.$$

Then we have

$$\begin{aligned} m_k(\hat{d}_k) - f(x_k) &= \phi_k^T \hat{d}_k + \frac{1}{2}(\hat{d}_k)^T H_k \hat{d}_k + \frac{\sqrt{\hat{h}_k(\delta_k)}}{2}\|\hat{d}_k\|^3 - \frac{\sqrt{\hat{h}_k(\delta_k)}}{6}\|\hat{d}_k\|^3 \\ &\leq \phi_k^T \hat{d}_k + \frac{1}{2}(\hat{d}_k)^T H_k \hat{d}_k + \frac{\gamma_k}{2}\|\hat{d}_k\|^2 - \frac{\sqrt{\hat{h}_k(\delta_k)}}{6}\|\hat{d}_k\|^3 \\ &= -\frac{1}{2}(\delta_k + \gamma_k) - \frac{\sqrt{\hat{h}_k(\delta_k)}}{6}\|\hat{d}_k\|^3 \leq -\frac{\sqrt{\hat{h}_k(\delta_k)}}{6}\|\hat{d}_k\|^3 \end{aligned} \tag{3.7}$$

Also, since $\hat{h}_k(\delta_k) \geq h_{\min}$, this completes the proof. $\square$

Lemma 3.2. *Suppose Assumption 3.1 holds, when the solution pair $[\hat{v}_k; \hat{t}_k]$ of the HSODF subproblem satisfies (3.5a)-(3.5b), the auxiliary function $\hat{h}_k(\delta)$ has a upper bound*

$$\hat{h}_k(\delta) \leq \varsigma_h := \max\left\{\hat{h}_0(\delta_0), 9\left(\varsigma_\phi + \frac{M}{2}\right)^2\right\}. \tag{3.8}$$

Whenever the value of $\hat{h}_k(\delta)$ reaches this upper bound, the iterates will be successful. As a result, we have

$$(\hat{t}_k)^2 \leq \frac{\varsigma_h}{\gamma_k^2 + \varsigma_h}. \tag{3.9}$$

Proof. Notice $f(x_k + \hat{d}_k) - f(x_k) = f(x_k + \hat{d}_k) - \hat{m}_k(\hat{d}_k) + \hat{m}_k(\hat{d}_k) - f(x_k)$, and

$$\begin{aligned}
\hat{m}_k(\hat{d}_k) - f(x_k + \hat{d}_k) &= (\phi_k - g_k)^T \hat{d}_k + \frac{1}{2}\hat{d}_k^{\,T}(H_k - \nabla^2 f(x_k + \xi \hat{d}_k))\hat{d}_k + \frac{\sqrt{\hat{h}_k(\delta_k)}}{3}\|\hat{d}_k\|^3 \\
&\geq -\varsigma_\phi \|\hat{d}_k\|^3 - \frac{M}{2}\|\hat{d}_k\|^3 + \frac{\sqrt{\hat{h}_k(\delta_k)}}{3}\|\hat{d}_k\|^3 \\
&= \left(\frac{\sqrt{\hat{h}_k(\delta_k)}}{3} - \frac{M}{2} - \varsigma_\phi\right)\|\hat{d}_k\|^3,
\end{aligned} \tag{3.10}$$

where $\xi \in [0,1]$. Therefore, $\hat{m}_k(\hat{d}_k) - f(x_k + \hat{d}_k) \geq 0$ holds as long as $\hat{h}_k(\delta_k) \geq 9(\frac{M}{2} + \varsigma_\phi)^2$. Then the ratio ρ_k follows,

$$\begin{aligned}
\rho_k = \frac{f(x_k) - f(x_{k+1})}{f(x_k) - m_k(\hat{d}_k)} &= \frac{f(x_k) - \hat{m}_k(\hat{d}_k) + m_k(\hat{d}_k) - f(x_{k+1})}{f(x_k) - m_k(\hat{d}_k)} \\
&= 1 + \frac{m_k(\hat{d}_k) - f(x_k + \hat{d}_k)}{f(x_k) - \hat{m}_k(\hat{d}_k)} \geq 1.
\end{aligned} \tag{3.11}$$

Hence iteration k must be very successful, and we get the desired bound. If $\hat{h}_k(\delta_k) \to \varsigma_h$ then $\rho_k \to 1$; i.e., it results in a successful iterate eventually. This implies (3.8). For (3.9), note that

$$\gamma_k^2 \frac{(\hat{t}_k)^2}{1 - (\hat{t}_k)^2} \leq \varsigma_h,$$

rearrange the items and we have the result. $\square$

Lemma 3.3. *Suppose that Assumption 3.1 holds, $\hat{d}_k$ is a successful step, and also the inexact solution pair satisfies (3.5a). Then we have*

$$\|\hat{d}_k\| \geq \Omega\left(\left\|\nabla f(x_k + \hat{d}_k)\right\|^{\frac{1}{2}}\right). \tag{3.12}$$

Proof. From the second-order Lipschitz continuity of f, we have

$$\begin{aligned}
\|\nabla f(x_k+\hat{d}_k)\| &\leq \|\nabla f(x_k+\hat{d}_k) - H_k\hat{d}_k - g_k\| + \|H_k\hat{d}_k+\phi_k\| + \|\phi_k - g_k\| \\
&\leq \frac{M}{2}\|\hat{d}_k\|^2 + \|H_k\hat{d}_k+\phi_k\| + \varsigma_\phi\|\hat{d}_k\|^2 \\
&\leq \frac{M}{2}\|\hat{d}_k\|^2 + \sqrt{\hat{h}_k(\delta_k)}\|\hat{d}_k\|^2 + \varsigma_\phi\|\hat{d}_k\|^2 + \varsigma_r\|\hat{d}_k\|^2 \\
&\leq \left(\frac{M}{2} + \sqrt{\varsigma_h} + \varsigma_\phi + \varsigma_r\right)\|\hat{d}_k\|^2.
\end{aligned} \tag{3.13}$$

The third inequality is from (3.5a). □

Next, suppose there exists negative curvature at the current iteration x_k, i.e., $\lambda_1(H_k) \leq \Omega(-\sqrt{\epsilon})$, we will show that the step can also bring enough decrease.

Lemma 3.4. *Suppose the inexact solution satisfies* (3.5c)*, then we have*

$$\|\hat{d}_k\| \geq -\frac{1}{2\sqrt{\varsigma_h}}\lambda_1(H_k). \tag{3.14}$$

Proof. Note that from (2.5c), we have

$$F_k + \gamma_k I + e_k I \succeq 0,$$

thus it follows that

$$\begin{aligned}
\gamma_k &= \sqrt{\hat{h}_k(\delta_k)}\|\hat{d}_k\| \\
&\geq -\lambda_{\min}(F_k) - e_k \\
&\geq -\lambda_{\min}(H_k) - e_k.
\end{aligned}$$

Combine it with (3.5c), we have

$$2\sqrt{\hat{h}_k(\delta_k)}\|\hat{d}_k\| \geq -\lambda_1(H_k).$$

□

The above lemmas show that an inexact step satisfying Condition 3.1 guarantees descent properties as in the exact case, answering questions (a). For a correspondence, one can compare the above lemmas with the exact case in [3, Lemma 3.1-Lemma 3.6]. The remaining analysis turns out to be the same as that of the exact case.

Consequently, under assumptions on the quality of Ritz pair (Condition 3.1), we arrive at the conclusion that the inexact adaptive HSODM has the same $O(\epsilon^{-3/2})$ iteration complexity as the exact version. We omit the proof since one simply replaces exact quantities with inexact ones provided above.

Theorem 3.1. *Suppose that the subproblem* (3.3) *is solved by the Lanczos method with an error tolerance $e_k \leq O(\epsilon^{1/2})$ and the approximated solution satisfies Condition 3.1. The adaptive HSODM takes $O\left(\epsilon^{-3/2}\right)$ iterations to achieve a point x_k satisfying $\|g_k\| \leq O(\epsilon)$ and $\lambda_1(H_k) \geq \Omega(-\sqrt{\epsilon})$.*

3.2 The complexity of bisection

The complexity rate cannot be established without the bisection method to locate δ_k. In this subsection, we briefly check that the bisection method still works when using $\hat{h}$ instead of h. We show the disparity between h and $\hat{h}$ can be measured by the quality of inexactness.

As we allow a tolerance of σ during the search procedure, we may connect the search interval $\hat{I}_h$ to the underlying exact I_h of h_k. We denote $\hat{I}_h : [\ell, \nu]$ as the target interval of $\hat{h}_k$.

Before we start, we made the following assumption.

Assumption 3.2. *For the homogeneous system (3.3), assume that there exists $\sigma_F > 0$ such that the following holds*

$$v^T F_k v - \lambda_1(F_k) \geq \sigma_F. \tag{3.15}$$

for some $\|v\| = 1$.

Note (3.15) holds without loss of generality if $\|F_k\| \gg e_k$; ad absurdum, the Lanczos almost terminates immediately. The results below are from [7]. For completeness, we also provide proof.

Lemma 3.5. *Suppose that $\sigma_F := \lambda_2(F_k) - \lambda_1(F_k) > 0$, then for the $[r_k; \sigma_k]$ from (2.5), we have*

$$\|[r_k; \sigma_k]\| \leq O(\sqrt{e_k/\sigma_F}). \tag{3.16}$$

and if $v_1 = [v_1; t_1] \in \mathcal{S}_1(F_k)$, then,

$$\|[\hat{v}_k; \hat{t}_k] - [v_1; t_1]\| \leq O(\sqrt{e_k/\sigma_F}). \tag{3.17}$$

Proof. From (2.5), we have

$$(\hat{d}_k)^T F_k \hat{d}_k + \gamma_k \|\hat{d}_k\|^2 = 0. \tag{3.18}$$

We can write $\hat{d}_k = \alpha v_1 + s$, where $s \perp v_1$. Since $\hat{d}_k$ is a unit vector, we have

$$\alpha^2 + \|s\|^2 = 1. \tag{3.19}$$

Then from (3.18) we have

$$\begin{aligned} -\theta_k + e_k \geq -\gamma_k &= (\hat{d}_k)^T F_k \hat{d}_k \\ &= -\theta_k \alpha^2 + s^T F_k s \\ &\geq -\theta_k \alpha^2 + (-\theta_k + \sigma_F)\|s\|^2. \end{aligned} \tag{3.20}$$

The second equality is obtained by expanding $\hat{d}_k$ and $s \perp v_1$. It implies

$$\|s\|^2 \leq \frac{e_k}{\sigma_F}. \tag{3.21}$$

Thus we have

$$\begin{aligned} r_k &= F_k\hat{d}_k + \gamma_k\hat{d}_k \\ &= (F_k + \gamma_k I)(\alpha v_1 + s) \\ &= \alpha(\gamma_k - \theta_k)v_1 + (F_k + \gamma_k I)s. \end{aligned} \tag{3.22}$$

Thus from (3.21), we can bound the norm of the residual:

$$\begin{aligned} \|r_k\| &\leq \alpha(\theta_k - \gamma_k) + \|(F_k + \gamma_k I)s\| \\ &\leq \alpha e_k + \|(F_k + \gamma_k I)\|\sqrt{\frac{e_k}{\sigma_F}} \\ &\leq \alpha e_k + 2U_F\sqrt{\frac{e_k}{\sigma_F}}. \end{aligned} \tag{3.23}$$

Where U_F denotes any upper bound of $\|F_k\|$. For the second part, note

$$\begin{aligned} \|\hat{d}_k - v_1\| &= \sqrt{\|\hat{d}_k - v_1\|^2} \\ &= \sqrt{\langle(1-\alpha) + s, (1-\alpha) + s\rangle} \\ &= \sqrt{2\|s\|^2} \leq O(\sqrt{e_k/\sigma_F}). \end{aligned} \tag{3.24}$$

□

We now rewrite the auxiliary function in terms of t_k,

$$\hat{h}_k(\delta_k) = \gamma_k^2 \cdot g(\hat{t}_k),\ h_k(\delta_k) = \theta_k^2 \cdot g(t_k) \tag{3.25}$$

and denote $g(t) = \frac{t^2}{1-t^2}$; note the upper bound of $\hat{t}_k$ has been found in Lemma 3.2. Now we locate $\hat{h}_k$ with a box interval of h_k as follows.

Lemma 3.6. *The auxiliary function $\hat{h}_k$ can be bounded both above and below by the function h_k in the following manner:*

$$h_k(\delta_k) - \hat{h}_k(\delta_k) \leq 2\frac{\varsigma_h}{\gamma_k}e_k + 2\sqrt{\varsigma_h}\frac{\left(\gamma_k^2 + \varsigma_h\right)^{3/2}}{\gamma_k^2}|t_k - \hat{t}_k| + o(|t_k - \hat{t}_k|). \tag{3.26}$$

and

$$\hat{h}_k(\delta_k) - h_k(\delta_k) \leq 2\sqrt{\varsigma_h}\frac{\left(\gamma_k^2 + \varsigma_h\right)^{3/2}}{\gamma_k^2}|t_k - \hat{t}_k| + o(|t_k - \hat{t}_k|). \tag{3.27}$$

Proof.

$$\begin{aligned} h_k(\delta_k) - \hat{h}_k(\delta_k) &= \theta_k^2 g(t_k) - \gamma_k^2 g(\hat{t}_k) \\ &= \theta_k^2 g(t_k) - \gamma_k^2 g(t_k) + \gamma_k^2 g(t_k) - \gamma_k^2 g(\hat{t}_k) \\ &= g(t_k)(\theta_k + \gamma_k)(\theta_k - \gamma_k) + \gamma_k^2 \frac{\mathsf{d}}{\mathsf{d}\hat{t}_k} g(\hat{t}_k)(t_k - \hat{t}_k) + o(|t_k - \hat{t}_k|) \\ &\leq 2g(t_k)\theta_k e_k + \gamma_k^2 \left|\frac{\mathsf{d}}{\mathsf{d}\hat{t}_k} g(\hat{t}_k)\right| |t_k - \hat{t}_k| + o(|t_k - \hat{t}_k|) \\ &\leq 2g(t_k)\theta_k e_k + \gamma_k^2 \left|\frac{\mathsf{d}}{\mathsf{d}\hat{t}_k} g(\hat{t}_k)\right| |t_k - \hat{t}_k| + o(|t_k - \hat{t}_k|). \end{aligned} \tag{3.28}$$

As for the part concerning $g'(\cdot)$, since $g'(\cdot)$ is monotonically increasing and $\hat{t}_k$ has an upper bound as in the previous analysis, we have

$$\frac{\mathsf{d}}{\mathsf{d}\hat{t}_k} g(\hat{t}_k) = \frac{2\hat{t}_k}{(1-(\hat{t}_k)^2)^2} \leq 2\sqrt{\varsigma_h}\left(\gamma_k^2+\varsigma_h\right)^{3/2}/\gamma_k^4.$$

Again with (3.28) we have

$$h_k(\delta_k) - \hat{h}_k(\delta_k) \leq 2\frac{\varsigma_h}{\gamma_k} e_k + 2\sqrt{\varsigma_h}\frac{\left(\gamma_k^2+\varsigma_h\right)^{3/2}}{\gamma_k^2}|t_k-\hat{t}_k| + o(|t_k-\hat{t}_k|).$$

Similarly, we have

$$\begin{aligned}
\hat{h}_k(\delta_k) - h_k(\delta_k) &= \gamma_k^2 g(\hat{t}_k) - \theta_k^2 g(t_k) \\
&= \gamma_k^2 g(\hat{t}_k) - \gamma_k^2 g(t_k) + \gamma_k^2 g(t_k) - \theta_k^2 g(t_k) \\
&\leq -\left[\gamma_k^2 \frac{\mathsf{d}}{\mathsf{d}\hat{t}_k} g(\hat{t}_k)(t_k-\hat{t}_k) + o\left(|t_k-\hat{t}_k|\right)\right] \\
&\leq \gamma_k^2 \left|\frac{\mathsf{d}}{\mathsf{d}\hat{t}_k} g(\hat{t}_k)\right| \cdot |t_k-\hat{t}_k| + o(|t_k-\hat{t}_k|) \\
&\leq 2\sqrt{\varsigma_h}\frac{\left(\gamma_k^2+\varsigma_h\right)^{3/2}}{\gamma_k^2}|t_k-\hat{t}_k| + o(|t_k-\hat{t}_k|).
\end{aligned}$$

□

To this end, the search procedure can be enabled if the inexactness of the Lanczos method is sufficiently small. Since now the search process is based on $\hat{h}_k$, denote the target interval $\hat{I}_h = [\ell, \nu]$, we have the following corollary.

Corollary 3.1. *Suppose that there exists an interval $\hat{I}_h = [\ell, \nu]$ such that $\hat{h}_k \in \hat{I}_h$, then the interval I_h below where $h_k \in I_h$ holds that*

$$I_h := \left[l + 2\frac{\varsigma_h}{\gamma_k}e_k + 2\sqrt{\varsigma_h}\frac{\left(\gamma_k^2+\varsigma_h\right)^{3/2}}{\gamma_k^2}|t_k-\hat{t}_k|, \nu - 2\sqrt{\varsigma_h}\frac{\left(\gamma_k^2+\varsigma_h\right)^{3/2}}{\gamma_k^2}|t_k-\hat{t}_k|\right].$$

Ignoring the high-order terms, a consequence of the above result is that if $\hat{h}_k(\delta_k) \notin \hat{I}_h$, then $h_k(\delta_k) \notin I_h$. As long as the length of the interval $\hat{I}_h$ is sufficiently large, the exact interval I_h will not be empty, and thus the bisection method is well-defined. Specifically, a nontrivial I_h is guaranteed by allowing σ as the length of $\hat{I}_h$.

Lemma 3.7. *Suppose (3.5) hold and the length of the interval $\hat{I}_h$ being σ, then the length of the interval I_h is at least $\frac{\sigma}{2}$.*

Proof. Note that when (3.5) and (3.32) hold, we have

$$2\sqrt{\varsigma_h}\frac{\left(\gamma_k^2+\varsigma_h\right)^{3/2}}{\gamma_k^2}|t_k-\hat{t}_k| \leq 2\sqrt{\varsigma_h}\frac{\left(\gamma_k^2+\varsigma_h\right)^{3/2}}{\gamma_k^2}\sqrt{\frac{e_k}{\sigma_F}} \leq \frac{1}{8}\sigma, 2\frac{\varsigma_h}{\gamma_k}e_k \leq \frac{1}{8}\sigma.$$

Combining the above facts with the length of $\hat{I}_h$ we have the proof. $\square$

Summarizing the above results, we have the following theorem.

Theorem 3.2. *At an iterate x_k, suppose the subproblem (3.3) is solved to satisfy Condition 3.1 and Condition 3.2 satisfied. Then the bisection method outputs some δ_k such that $\hat{h}_k(\delta_k) \in \hat{I}_h$ within tolerance σ in at most*

$$O\left(\log\left(\frac{\varsigma_h U_\phi U_H}{h_{\min}\sigma}\right)\right) \tag{3.29}$$

iterations.

Proof. Combining all the above results, we notice that placing $h_k \in I_h$ can be done implicitly by trying $\hat{h}_k(\delta_k) \in \hat{I}_h$. We let the bisection proceed if $\hat{h}_k(\delta_k) \notin \hat{I}_h$, which implies $h_k(\delta_k) \notin I_h$. As $|I_h| > \frac{\sigma}{2}$ is guaranteed from a $\hat{I}_h$ of σ, the bisection method has the same number of arithmetic operations as in the exact case. $\square$

3.3 Dealing with the hard case

Different from the method with exact eigenvalues, the inexactness of bisection procedure intertwines with the hard case. When $\hat{t}_k = 0$, the inexact hard case follows from (2.5a)-(2.5d),

$$(H_k + \gamma_k I)\,\hat{v}_k = r_k \tag{3.30a}$$

$$\hat{v}_k \perp r_k, \phi_k^T \hat{v}_k = \sigma_k \tag{3.30b}$$

$$F_k + \gamma_k I + e_k I \succeq 0, \tag{3.30c}$$

where $\hat{t}_k = 0$ no longer indicates that we get the exact value of $\lambda_1(H_k)$. We follow the same perturbation treatment as in [3, Algorithm 3], while using the Ritz pairs. For completeness, we present the method in Algorithm 2. We denote k as the current iterate, then the subsequent iterates are denoted by i: $k, k+1, ..., k+i$ and so forth.

Assume that e_k satisfies the following condition.

Condition 3.2. *Suppose the Lanczos method is performed until the inexactness e_k satisfies*

$$e_k \le \min\left\{\frac{\gamma_k \sigma}{16\varsigma_h}, \frac{\sigma^2 \sigma_F}{256} \frac{\gamma_k^4}{(\gamma_k^2 + \varsigma_h)^3}\right\}. \tag{3.32}$$

While $\hat{t}_k = 0$ no longer induces the eigenpair of H_k, if r_k, σ_k is tolerable, $\gamma_k, \hat{v}_k$ still serves as an approximation to the leftmost eigenpair of H_k.

Lemma 3.8. *Suppose that $\hat{t}_k = 0$ occurs, we have*

$$\gamma_k \le -\lambda_1(H_k). \tag{3.33}$$

Algorithm 2: Perturbation for the Hard Case

Input: Iterate k, $x_k \in \mathbb{R}^n$, g_k, H_k, $\hat{h}_{k-1}$, δ_{k-1} where $g_k \perp \mathcal{S}_1$;

for $i = 0, 1, \dots$ **do**

Set

$$\phi_{k+1} = \phi_k + \left(\operatorname{sign}(\sigma_k)\frac{\varsigma_\phi \gamma_k^2}{4\nu}\right) \cdot \hat{v}_k. \tag{3.31}$$

Compute $I_h := [\iota_2\sqrt{\hat{h}_{k+i-1}}, \iota_3\sqrt{\hat{h}_{k+i-1}}]$;

repeat `// inner iterates` j `via bisection (see` **??**`)`

Obtain the solution $[\hat{v}_{k+i,j}; \hat{t}_{k+i,j}]$ of the GHM subproblem

$$\min_{\|[v;t]\|\le 1} \begin{bmatrix} v \\ t \end{bmatrix}^T \begin{bmatrix} \hat{h}_k & \phi_{k+i} \\ (\phi_{k+i})^T & \delta_{k+i,j} \end{bmatrix} \begin{bmatrix} v \\ t \end{bmatrix}$$

set $\hat{d}_{k+i,j} = \hat{v}_{k+i,j}/\hat{t}_{k+i,j}$, $\hat{h}_{k+i} := \left(\theta_{k+i,j}/\|\hat{d}_{k+i,j}\|\right)^2$;

Update $\delta_{k+i,j}$, increase $j = j + 1$

until $\sqrt{\hat{h}_{k+i}} \in I_h$ *within tolerance* σ;

$\hat{d}_{k+i} = \hat{d}_{k+i,j}$; $\delta_{k+i} = \delta_{k+i,j}$;

Compute

$$\rho_{k,i} = \frac{f(x_k + \hat{d}_{k+i}) - f(x_k)}{m_k(\hat{d}_{k+i}) - f(x_k)}$$

if $\rho_{k+i} \ge \iota_1$ **then**

break

end

end

Moreover, suppose that $\hat{v}_k = \alpha v_1 + s$, *where* v_1 *is the eigenvector correspond to* $\lambda_1(H_k)$, $s \perp v_1$, *we have*

$$\alpha \ge \sqrt{1 - \frac{e_k}{\sigma_H}}. \tag{3.34}$$

$\sigma_H := \lambda_2(H_k) - \lambda_1(H_k)$.

Proof. Multiply both sides of (3.30a) by $\hat{v}_k$ and note that $\hat{v}_k \perp r_k$, we have

$$\hat{v}_k^T H_k \hat{v}_k = -\gamma_k \|\hat{v}_k\|^2,$$

which implies (3.33). To verify (3.34), just note that the previous lemma just tells us that

$$\|s\| \le \sqrt{\frac{e_k}{\sigma_H}},$$

combine it with $\alpha^2 + \|s\|^2 = 1$ and we have (3.34). $\square$

In the exact case, for each of the following iterate i, we perturb ϕ_{k+i} based on the preceding h_{k+i-1}. When gradually increasing h_{k+i}, we use the same bisection method indexed by j to find $\delta_{k+i,j}$. We show that it will finally produce a *successful iteration*; once it does, it must satisfy the Assumption 3.1. We show these goals can be achieved via Ritz vectors.

Lemma 3.9. *Suppose at the k-th iterate $\hat{t}_k = 0$ and the search interval of $\hat{h}_{k+1}$ is $[\ell, \nu]$. Then if ϕ_k is set according to (3.31), and inexactness measure e_{k+1} satisfy*

$$e_{k+1} \leq e_k \leq \frac{-2\|\phi_k\| + \sqrt{4\|\phi_k\|^2 + \frac{\varsigma_\phi \gamma_k^2}{\nu}\left(\|\phi_k\| + \frac{\varsigma_\phi \gamma_k^2}{2\nu}\right)}}{2\|\phi_k\| + \frac{\varsigma_\phi \gamma_k^2}{\nu}}, \tag{3.35}$$

then the subsequent $t_{k+1} \neq 0$, i.e., the hard case is eliminated.

Proof. We prove by contradiction, suppose that $t_{k+1} = 0$, denote $\hat{v}_{k+1} = \alpha_1 v_1 + s_1$, by Lemma 3.8, we have $\alpha_1 \geq \sqrt{1 - \frac{e_{k+1}}{\sigma_H}}$. Then we have

$$\begin{aligned}
\hat{v}_{k+1}^T \phi_{k+1} &= \left(\phi_k + \operatorname{sign}(\sigma_k)\frac{\varsigma_\phi \gamma_k^2}{4\nu}\hat{v}_k\right)^T \hat{v}_{k+1} \\
&= \phi_k^T \hat{v}_{k+1} + \operatorname{sign}(\sigma_k)\frac{\varsigma_\phi \gamma_k^2}{4\nu}\hat{v}_k^T \hat{v}_{k+1} \\
&= \phi_k^T \hat{v}_k + \phi_k^T(\hat{v}_{k+1} - \hat{v}_k) + \operatorname{sign}(\sigma_k)\frac{\varsigma_\phi \gamma_k^2}{4\nu}\hat{v}_k^T \hat{v}_{k+1} \\
&= \sigma_k + \phi_k^T(\hat{v}_{k+1} - \hat{v}_k) + \operatorname{sign}(\sigma_k)\frac{\varsigma_\phi \gamma_k^2}{4\nu}\hat{v}_k^T \hat{v}_{k+1} \\
&= \sigma_k + \phi_k^T(\alpha_1 v_1 + s_1 - \alpha v_1 - s) + \operatorname{sign}(\sigma_k)\frac{\varsigma_\phi \gamma_k^2}{4\nu}(\alpha\alpha_1 + s^T s_1).
\end{aligned}$$

In the last line, we plug in $\hat{v}_{k+1} = \alpha_1 v_1 + s_1$ and $\hat{v}_k = \alpha v_1 + s$. Using the fact that $e_{k+1} \leq e_k$, we have

$$\begin{aligned}
|\hat{v}_{k+1}^T \phi_{k+1}| &\geq \sigma_k + \frac{\varsigma_\phi \gamma_k^2}{4\nu}\left(\sqrt{1 - \frac{e_{k+1}}{\sigma_H}}\sqrt{1 - \frac{e_k}{\sigma_H}} - \|s\|\|s_1\|\right) - \|\phi_k\| \left\|(\alpha_1 - \alpha) v_1 + s_1 - s\right\| \\
&\geq \sigma_k + \frac{\varsigma_\phi \gamma_k^2}{4\nu}\left(\sqrt{1 - \frac{e_{k+1}}{\sigma_H}}\sqrt{1 - \frac{e_k}{\sigma_H}} - \sqrt{\frac{e_k}{\sigma_H}}\sqrt{\frac{e_{k+1}}{\sigma_H}}\right) - \|\phi_k\| \left(|\alpha_1 - \alpha| + \|s_1\| + \|s\|\right) \\
&\geq \sigma_k + \frac{\varsigma_\phi \gamma_k^2}{4\nu}\left(1 - 2\frac{e_k}{\sigma_H}\right) - \|\phi_k\| \left(1 - \sqrt{1 - \frac{e_k}{\sigma_H}} + 2\sqrt{\frac{e_k}{\sigma_H}}\right) \\
&> \sigma_k + \frac{\varsigma_\phi \gamma_k^2}{4\nu}\left(1 - 2\frac{e_k}{\sigma_H}\right) - \|\phi_k\| \left(\frac{e_k}{\sigma_H} + 2\sqrt{\frac{e_k}{\sigma_H}}\right).
\end{aligned}$$

The last inequality is because of $1 - \sqrt{1 - x^2} \leq x^2$ for $0 < x < 1$. Thus from (3.35) we know that

$$|\hat{v}_{k+1}^T \phi_{k+1}| > \sigma_k,$$

which contradicts $t_{k+1} = 0$. $\square$

We now show Algorithm 2 gradually produces ϕ_{k+1} that satisfies Assumption 3.1.

Lemma 3.10. *Suppose at the k-th iteration the hard case occurs, we perturb the gradient as in the way (3.31) and find δ_{k+1} such that $\hat{h}_{k+1}(\delta_{k+1}) \in [\ell, \nu]$ with $e_{k+1} \leq \sqrt{h_{\min}}\|\hat{d}_{k+1}\|$, then at the $(k+1)$-th iteration, the following relation holds*

$$\|g_{k+1} - \phi_{k+1}\| \leq \kappa_\phi \|\hat{d}_{k+1}\|^2.$$

Proof. We have already shown that $t_{k+1} \neq 0$ in Lemma 3.9, thus d_{k+1} is well-defined, from (3.30c) we know that

$$\sqrt{\hat{h}_{k+1}(\delta_{k+1})}\|\hat{d}_{k+1}\| \geq -\lambda_1(F_{k+1}) - e_{k+1} \geq -\lambda_1(H_k) - e_{k+1}.$$

Rearrange items we have

$$\|\hat{d}_{k+1}\| \geq -\frac{\lambda_1(H_k)}{2\nu} \geq -\frac{\gamma_k}{2\nu},$$

since we do not update when a hard case occurs,

$$\|g_{k+1} - \phi_{k+1}\| = \|g_k - \phi_{k+1}\| = \frac{\varsigma_\phi \gamma_k^2}{4\nu} \varsigma_\phi \|\hat{d}_{k+1}\|^2.$$

□

The rest is to show when the hard case occurs, the Algorithm 2 eventually produces a successful iterate. The proof is same as in the original paper [3, Theorem 3.5], we omit it here.

Theorem 3.3. *Algorithm 2 takes at most $\lfloor \log_{\iota_2} \frac{\varsigma_h}{\hat{h}_{k-1}(\delta_{k-1})} \rfloor + 1$ iterations to obtain a successful step. Furthermore, Assumption 3.1 must hold.*

4 Conclusion

In this report, we discuss the complexity analysis of the inexact version of adaptive HSODM [3, Algorithm 2]. The method allows inexact solutions in GHMs by the Lanzcos method. We discuss the convergence analysis, bisection method, and the hard case in the inexact setting based on Ritz vectors. All of the necessary elements in the exact case are replaced by their inexact counterparts. We validate that the inexact adaptive HSODM has the same iteration complexity as the exact version.

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